Volterra network modeling of the nonlinear finite-impulse reponse of the radiation belt flux

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Abstract. We show how a general class of spatio-temporal nonlinear impulse-response forecast networks (Volterra networks) can be constructed from a taxonomy of nonlinear autoregressive integrated moving average with exogenous inputs (NARMAX) input-output equations, and used to model the evolution of energetic particle f uxes in the Van Allen radiation belts. We present initial results for the nonlinear response of the radiation belts to conditions a month earlier. The essential features of spatio-temporal observations are recovered with the model echoing the results of state space models and linear f nite impulse-response models whereby the strongest coupling peak occurs in the preceding 1-2 days. It appears that such networks hold promise for the development of accurate and fully data-driven space weather modelling, monitoring and forecast tools.

Keywords: Radiation belts, input-output models, nonlinear neural networks
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1. INTRODUCTION

The chain of events leading to geospace magnetic storms begins with the ejection of solar plasma and plasma waves, followed by their propagation through the interplanetary medium and subsequent impact on the Earth's magnetosphere. Magnetic storms and substorms, two of the major complex dynamic phenomena in the terrestrial magnetosphere, have a number of distinct effects on the geospace environment such as electron and ion acceleration, auroral displays in the upper atmosphere at high latitudes, and geomagnetically-induced currents on the ground which may lead to electrical grid blackouts[5, 7, 6]. A highly prominent effect is the energization of the Van Allen radiation belts (see for example [3]) which is of particular interest to satellite operators, since radiation belt enhancements endanger spacecraft circuits and subsystems.

Many efforts in the last two decades have focused on the development of data-derived dynamical models of the energetic particle f ux in the radiation belts. Early linear prediction f lter studies focused on the temporal response of daily-averaged relativistic electrons at geostationary altitudes to solar plasma, interplanetary and magnetospheric drivers[10, 1]. Vassiliadis et al[14] extended this technique spatially by incorporating SAMPEX/PET electron f ux data over a broad range of L-shells from 1.1 to 10 Earth Radii (RE). The f rst self-consistent spatio-temporal state space models were then constructed[11], providing "one-step ahead" predictions for the nonlinear impulse-reponse of the radiation belts and, in particular, provided new and important clues to the timescales (typically days) involved in the dissipation of f ux.

In order to attempt to increase accuracy and, more importantly, to extend forecasts further forward in time in preparation for the development of a radiation belt storm warning index, we have adopted a methodology based on assuming an equivalence between NARMAX input-output equations and time-delay neural networks (which we will refer to as Volterra networks). Key to the success of this method are three fundamental theorems: the Wold Theorem[17] for general time series decomposition of data, Takens' Theorem[13] for time-delay embedding of nonlinear dynamics, and Hornik's Theorem[8] for universal function approximator neural networks. Takens' Theorem postulates that nonlinear input-output equations for decomposed time series data are fully capable of representing the nonlinear dynamics of the system under study provided that enough time lag variables are used. Equivalent Volterra networks can then be constructed by using time delays on the inputs and outputs and training on time series data to identify the nonlinear functional relation between input and output variables.

In this paper we brief y outline the method and present initial results of a 30^{th} order auto-regressive (feedback) 2D spatio-temporal model of the electric f ux in the radiation belts and show how the nonlinear f nite-impulse response transfer function as a function of lag time can be extracted from the network.

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2. METHODOLOGY

A schematic of the overall methodology adopted here is presented in Figure 1.

STEP 1: Construction of a taxonomy of NARMAX input-output equations

We began with a generalisation of the Wold time series decomposition [17] having the form,

$$J(t) = c + \Psi(t) + \varepsilon(t) \equiv \hat{J}(t) + \varepsilon(t) \tag{1}$$

where $J(t)$ is the electron f ux time series, $\hat{J}(t) = c + \Psi(t)$ are the model predictions, c is a constant (zero in the absence of trend), $\varepsilon(t) = J(t) - \hat{J}(t)$ are the prediction errors and $\Psi(t)$ is an "information function" constructed from lag series Φ_p, Θ_q, Ω_r

$$\Phi_p J(t) = \sum_{i=1}^{p} \varphi_i f_i \left[L^i J(t)\right] \tag{2}$$

$$\Theta_q \varepsilon(t) = \sum_{j=1}^{q} \theta_j g_j \left[L^j \varepsilon(t)\right] \tag{3}$$

$$\Omega_r I(t) = \sum_{k=1}^{r} \omega_k h_k \left[L^k I(t)\right] \tag{4}$$

with coeff cients φ_i, θ_j, ω_k, general functions f_i, g_j, h_k and lag operators,

$$L^i J(t) = J(t-i) \tag{5}$$

$$L^j \varepsilon(t) = \varepsilon(t-j) \tag{6}$$

$$L^k I(t) = I(t-k), \tag{7}$$

such that,

$$J(t) = c + \sum_{i=1}^{p} \varphi_i f_i \left[J(t-i)\right] + \sum_{j=1}^{q} \theta_j g_j \left[\varepsilon(t-j)\right] + \sum_{k=1}^{r} \omega_k h_k \left[I(t-k)\right] + \varepsilon(t) \tag{8}$$

in accordance with the Nonlinear AutoRegressive Moving-Average eXogenous input NARMAX(p,q,r) process. The introduction of general functions f, g and h into the lag series allows for a generalisation of the polynomial NARMAX models of Leontaritis and Billings (1985)[4]. Furthermore, in the case that the nonlinear system is driven by several inputs s then we will have a vector of inputs $I_l = \mathbf{I} = [I_1, I_2, \ldots, I_s]^\dagger$ and a corresponding vector of input lag series $\Omega_{r_l} = \Omega_\mathbf{r} = [\Omega_{r_1}, \Omega_{r_2}, \ldots, \Omega_{r_s}]$ where $r_l = \mathbf{r} = [r_1, r_2, \ldots, r_s]$ so that each I_l can have its own lag order r_l. In this general case, the nonlinear time series decomposition will be given by,

$$J(t) = c + \Phi_p J(t) + \Theta_q \varepsilon(t) + \Omega_\mathbf{r} \bullet \mathbf{I}(t) + \varepsilon(t) \tag{9}$$

and represents the general NARMAX$(p,q,\mathbf{r})$ process. Note that, our notation allows the NARMAX process to be written as an *explicit* sum of terms rather than the traditional presentation which would describe, for example, the NARMAX(p,q,r) process in terms of a general polynomial function F as follows, follows[4],

$$\begin{aligned} J(t) = c \quad + \quad & F\{J(t-1), J(t-2)\ldots, J(t-p); \\ & \varepsilon(t-1), \varepsilon(t-2), \ldots, \varepsilon(t-q); \\ & I(t-1), I(t-2), \ldots, I(t-r)\} + \varepsilon(t). \end{aligned}$$

It is precisely the linear nature of the explicit sum introduced above that permits the development of a connectionist solution via neural network architectures. To recap, the information function contains linear combinations of (general nonlinear) operators acting on the autoregressive time-delayed (lagged) time series of radiation belt f ux $J(t-p)$, moving-average lagged equation errors $\varepsilon(t-q)$, and lagged exogenous inputs $\mathbf{I}(t-\mathbf{r})$. The particular class of model chosen depends on how exactly $\Psi(t)$ is def ned from the form of the functions f_i, g_j and h_k and the order of the autoregression, the moving average and the exogenous inputs. Table 1 below shows some key examples from the general taxonomy of NARMAX$(p,q,\mathbf{r})$ input-output equations.

STEP 2: Inclusion of nonlinear dynamics via time-delay embedding

Since Takens' Theorem[13] means that there is a 1-to-1 mapping between a time series and the underlying dynamical state space, then, provided that a nonlinear functional is used and enough lag variables are incorporated into the specif cation of the input-output model, equivalence will exist between any given NARMAX$(p,q,\mathbf{r})$ process and the nonlinear dynamical system it aims to represent. Assuming for now that the model orders p, q and $\mathbf{r}$ can be identif ed using phase space techniques (such as false nearest neighbours) then the problem at hand reduces to solving the follow equation for the nonlinear time series decomposition of the radiation belt f ux $J(t)$ in terms of the unknown coeff cients ϕ_i, ϕ_j, $\omega_{l,k}$ and the functions f_i, g_j and h_l, k,

$$\begin{aligned} J(t) = c \quad + \quad & \varphi_1 f_1 \left[J(t-1)\right] + \cdots + \varphi_p f_p \left[J(t-p)\right] \\ + \quad & \theta_1 g_1 \left[\varepsilon(t-1)\right] + \cdots + \theta_q g_q \left[\varepsilon(t-q)\right] \\ + \quad & \omega_{1,1} h_{1,1} \left[I_1(t-1)\right] + \cdots \end{aligned}$$

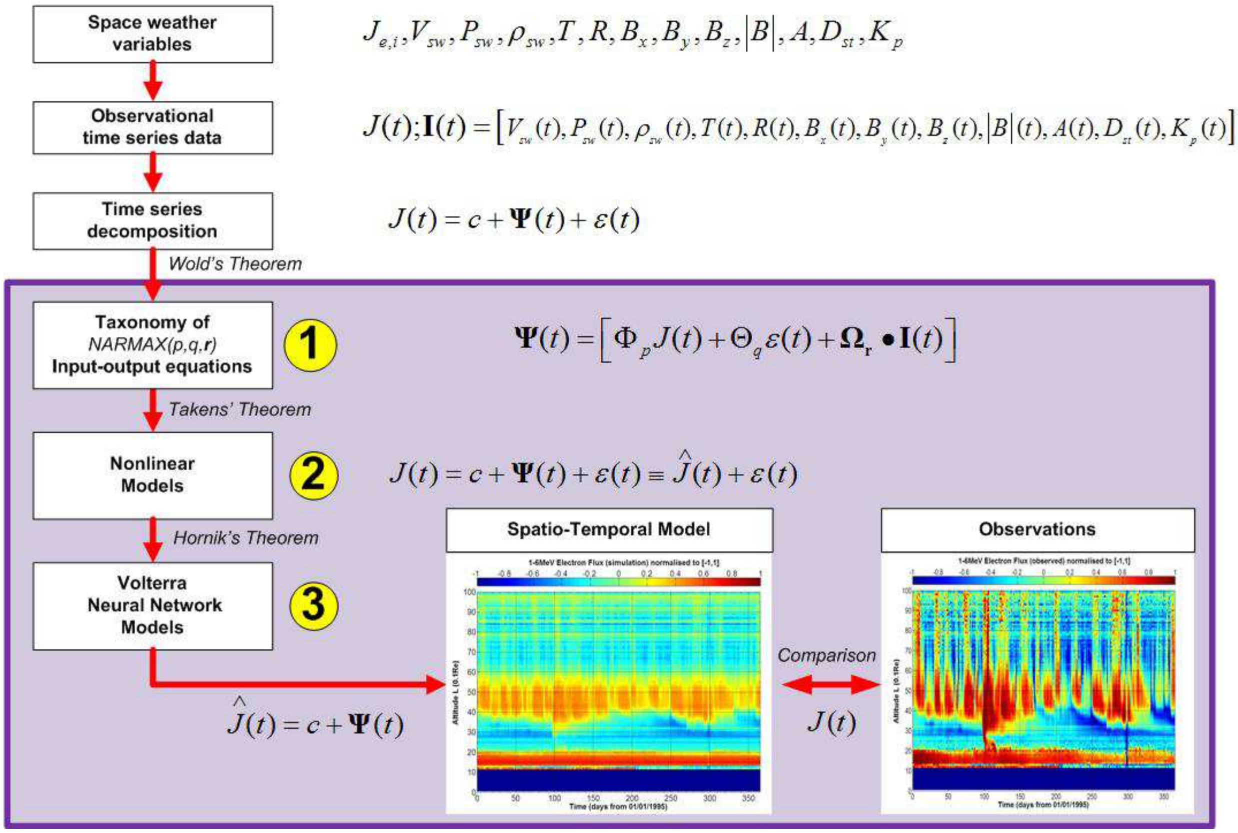

FIGURE 1. The overall methodology we have adopted together with a comparison of the observed SAMPEX/PET electron f ux observations normalised to the interval [-1,1] with initial results from our 2D spatio-temporal model based on the nonlinear autoregressive *NAR*(30) Volterra network.

TABLE 1. A taxonomy of input-ouput models in the NARMAX$(p,q,\mathbf{r})$ class

Functions f, g, h	Autoregression Order	Moving-Avergae Order	Inputs	Model
1	1	0	0	AR(1)=Random walk
1	p	0	0	AR(p)
1	p	0	r	ARX(p,r)
1	p	0	$\mathbf{r}$	ARX$(p,\mathbf{r})$
1	0	q	0	MA(q)
1	0	∞	0	MA(∞)=Wold Decomposition
1	0	q	r	MAX(q,r)
1	0	q	$\mathbf{r}$	MAX$(q,\mathbf{r})$ (multivariate)
1	p	q	0	ARMA(p,q)
1	p	q	r	ARMAX(p,q,r)
1	p	q	$\mathbf{r}$	ARMAX$(p,q,\mathbf{r})$ (multivariate)
f	p	0	0	NAR(p)
f,h	p	0	r	NARX(p,r)
f,h	p	0	$\mathbf{r}$	NARX$(p,\mathbf{r})$ (multivariate)
g	0	q	0	NMA(q)
g,h	0	q	r	NMAX(q,r)
g,h	0	q	$\mathbf{r}$	NMAX$(q,\mathbf{r})$ (multivariate)
f,g	p	q	0	NARMA(p,q)
f,g,h	p	q	r	NARMAX(p,q,r)
f,g,h	p	q	$\mathbf{r}$	NARMAX$(p,q,\mathbf{r})$ (multivariate)